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On the Rumin complex

Akahori, Takao (赤堀　隆夫)

The moduli of complex structures seems to be related to the hyperbolicity. Here, we treat the moduli of isolated singularities. Let (V, o) be a normal isolated singularity in a complex euclidean space C^N. Consider the link M, defined by the intersection of (V, o) and the real hypersphere $S_\epsilon^{2N-1}(o)$, centered at the origin o with the radius of ϵ. Over this link M, a CR structure is naturally induced from V. Concerning the moduli problem of isolated singularities, Kuranishi initiated the deformation theory of CR structures. Of course, the problem of constructing a versal family of deformations of isolated singularities is done by several authors, almost 30 years ago(for example, Grauert,Douady,etc..),with more direct method(rather, in algebraic geometry). However, our approach(in CR geometry) has one geometric aspect. In order to introduce "several new methods in topology (Symplectic structure, Seiberg-Witten invariants,etc..)" to the moduli of isolated singularities, our method is definitely more accessible(we are directly treating links, contact structures and CR structures).

The purpose of this survey is to introduce the brief sketch of the Rumin's method in the deformation theory of CR structures and establish a versal family in the 5-dimensional case(the precise proof will be published elsewhere). Obviously, the moduli of CR structures should be related to "Seiberg-Witten equation in CR strucures". In the future, I wolud like to discuss the relation of our setting and "Seiberg-Witten equatin in CR structures". But, at the present time, our project in this direction is in process. Here, we only sketch our joint work([A-G-L[1]],[A-G-L[2]]), briefly. Anyway, we start with the deformation theory of CR structures.

1 Standard Deformation Complex

Let $(M, {}^0T'')$ be a CR structure. This means that M is a C^∞ differentiable manifold with real odd dimension and ${}^0T'$ is a sub-vector bundle of the complexfied tangent bundle $C \otimes TM$ satisfying:

$$
{}^0T'' \cap {}^0T' = 0, dim_C \frac{C \otimes TM}{{}^0T'' + {}^0T'} = 1 \tag{1}
$$

$$
[\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''), \tag{2}
$$

where ${}^0T' = \overline{{}^0T''}$.

Now we assume that there is a global real vector field ξ satisfying; for every point p of M,

$$
\xi_p \notin {}^0T''_p + {}^0T'_p. \tag{3}
$$

We set

$$
T' := {}^0T' + C \otimes \xi, \tag{4}
$$

where $C \otimes \xi$ means the line bundle generated by ξ. For brevity, we use the notation F for this line bundle. By using this T', we set a C^∞ vector bundle decomposition

$$
C \otimes TM = T' + {}^0T''. \tag{5}
$$

Following [A1], we introduce a first order differential operator $\overline{\partial}_{T'}$ from $\Gamma(M, T')$ to $\Gamma(M, T' \otimes ({}^0T'')^*)(\Gamma(M, T')$(resp. $\Gamma(M, T' \otimes ({}^0T'')^*)$) means the space of C^∞ sections of T'(resp. $T' \otimes ({}^0T'')^*$), by: for $u \in \Gamma(M, T')$, $X \in {}^0T''_p$, p is a point of M,

$$
\overline{\partial}_{T'} u(X) = [X, u]_{T'}, \tag{6}
$$

where $[X, u]_{T'}$ means the projection of $[\tilde{X}, u]$ to T' according to (5), and $\tilde{X}$ means a C^∞ extension of X to M($\overline{\partial}_{T'} u(X)$ doesn't depend on the extension). Like the case for scalar valued differential forms, we can introduce $\overline{\partial}^{(p)}_{T'}$ operators, $p = 1, 2, \cdots$ (see [A1]) and we have a differential complex

$$
\begin{aligned}
0 \to \Gamma(M, T') &\overset{\overline{\partial}_{T'}}{\to} \Gamma(M, T' \otimes ({}^0T'')^*) \overset{\overline{\partial}^{(1)}_{T'}}{\to} \Gamma(M, T' \otimes \wedge^2({}^0T'')^*) \to \\
&\to \Gamma(M, T' \otimes \wedge^p({}^0T'')^*) \overset{\overline{\partial}^{(p)}_{T'}}{\to} \Gamma(M, T' \otimes \wedge^{p+1}({}^0T'')^*) \to \cdots
\end{aligned} \tag{7}
$$

with $\overline{\partial}^{(p+1)}_{T'} \overline{\partial}^{(p)}_{T'} = 0$.

This complex is called the standard deformation complex. We, briefly recall the deformation theory of CR structures. Let $(M, {}^0T'')$ be a CR structure.

Definition 1 *Let E be a complex subbundle of the complexfied tangent bundle $C \otimes TM$ satisfying:*

$$E \cap \overline{E} = 0.$$

We call this pair (M, E) an almost CR structure.

Almost CR structures satisfying a certain condition can be parametrized by elements of $\Gamma(M, T' \otimes ({}^0T'')^*)$ as follows.

Definition 2 *An almost CR structure (M, E) is at finite distance from $(M, {}^0T'')$ if and only if the composition map of the inclusion map of E into $C \otimes TM$, and the projection map of $C \otimes TM$ to ${}^0T''$ according to (5), is isomorphic.*

Then we have

Theorem 1 *(Proposition 1.1 in [A]). The almost CR structure, which is at finite distance from $(M, {}^0T'')$, corresponds to an element ϕ in $\Gamma(M, Hom({}^0T'', T'))$ $= \Gamma(M, T' \otimes ({}^0T'')^*)$, bijectively. The correspondence is that; for ϕ in $\Gamma(M, T' \otimes ({}^0T'')^*)$,*

$${}^\phi T'' = \{\ X'\ :\ X' = X + \phi(X),\ X \in {}^0T''\ \}.$$

And the following theorem explains when this almost CR structure $(M, {}^\phi T'')$ is really a CR structure.

Theorem 2 *(Theorem 2.1 in [A]). An almost CR structure $(M, {}^\phi T'')$, which is at finite distance from $(M, {}^0T'')$ is a CR structure(this means that our $(M, {}^\phi T'')$ satisfies the integrability condition), if and only if our ϕ is a solution of the non linear partial differential equations*

$$\begin{aligned} P(\phi) &= \overline{\partial}_{T'}^{(1)} \phi + R_2(\phi) + R_3(\phi) \\ &= 0. \end{aligned}$$

Namely the linear term of $P(\phi) = 0$ is $\overline{\partial}_{T'}^{(1)} \phi$. And the real difficult problem, in soloving this non-linear partial differential equation, is that: the non-linear term $R_2(\phi)$ includes the first order derivatives of ϕ, and our $\overline{\partial}_{T'}$ cpmplex is subelliptic(the Neumann operator gains only 1). So, in CR case, the mehod in complex manifolds is not available. In order to overcome this difficulty, we introduce E_j structure in [A1]. Henceforth, we assume strongly convexity.

2 E_j structure

We recall several results obtained in [A1]. Let $(M, {}^0T'')$ be a strongly speudo convex CR structure(this is an abstract of my talk, so we omit the notion of strongly pseudo convex CR structures). We set

$$\Gamma_j = \{u : u \in \Gamma(M, {}^0T' \otimes \wedge^j({}^0T'')^*), (\overline{\partial}^{(j)}_{T'} u)_{F\otimes\wedge^{j+1}({}^0T'')^*} = 0\} \tag{8}$$

Here $(\overline{\partial}^{(j)}_{T'} u)_{F\otimes\wedge^{j+1}({}^0T'')^*}$ means the projection of $\overline{\partial}^{(j)}_{T'} u$ to $F \otimes \wedge^j({}^0T'')^*$ according to the C^∞ vector bundle decomoposition

$$T' \otimes \wedge^{j+1}({}^0T'')^* = {}^0T' \otimes \wedge^{j+1}({}^0T'')^* + F \otimes \wedge^{j+1}({}^0T'')^*$$

induced by (4) and (5). In the definition of Γ_j, the first order derivative of u appears. But, actually as u takes its value in ${}^0T'$, $(\overline{\partial}^{(j)}_{T'} u)_{F\otimes\wedge^{j+1}({}^0T'')^*} = 0$ is an algebraic condition. We see this more precisely. We consider a bundle map:

$${}^0T' \otimes \wedge^j({}^0T'')^* \to F \otimes \wedge^{j+1}({}^0T'')^*$$

defined by; for u in ${}^0T' \otimes \wedge^j({}^0T'')^*$,

$$(\overline{\partial}^{(j)}_{T'} u)_{F\otimes\wedge^{j+1}({}^0T'')^*}.$$

Then, we have the following theorems.

Theorem 3 *(Proposition 2.1 in [A1]). There is a sub-vector bundle E_j of ${}^0T' \otimes \wedge^j({}^0T'')^*$, satisfying $\Gamma(M, E_j) = \Gamma_j$. And especially, our CR structure is strongly pseudo convex, $E_0 = 0$.*

Theorem 4 *(Theorem 2.2 in [A1]).*

$$\overline{\partial}^{(j)}_{T'} \Gamma_j \subset \Gamma_{j+1},$$

that is to say, $(\Gamma_j, \overline{\partial}_j)$, where

$$\overline{\partial}_j = \overline{\partial}^{(j)}_{T'} \mid_{\Gamma_j}$$

is a sub-differential complex of the standard deformation complex of $(\Gamma(M, T' \otimes \wedge^j({}^0T'')^*), \overline{\partial}^{(j)}_{T'})$.

So we have a sub-differential complex of the standard deformation complex;

$$0 \to 0 \to \Gamma(M, E_1) \xrightarrow{\overline{\partial}_1} \Gamma(M, E_2) \xrightarrow{\overline{\partial}_2} \Gamma(M, E_3). \tag{9}$$

And for this complex, we have

Theorem 5 *(Theorem 2.4 in [A1]). This subcomplex* $(\Gamma(M, E_j), \overline{\partial}_j)$ *recovers the cohomology group of the standard deformation complex. More precisely, the inclusion map induces the isomorphim map*

$$\frac{Ker\ \overline{\partial}_j}{Im\ \overline{\partial}_{j-1}} \simeq \frac{Ker\ \overline{\partial}_{T'}^{(j)}}{Im\ \overline{\partial}_{T'}^{(j-1)}}\ \ if\ j \geq 2,$$

and if $j = 1$, *the inclusion map induces the surjecive map*

$$Ker\ \overline{\partial}_1 \to \frac{Ker\ \overline{\partial}_{T'}^{(1)}}{Im\ \overline{\partial}_{T'}} \to 0.$$

Furthermore for this complex,

Theorem 6 *(Theorem 4.1 in [A1]) If* $dim_{\mathbf{R}} M = 2n - 1 \geq 7$, *then at* E_2, *there is a sub- elliptic estimate. So this estimate insures the Kodaira-Hodge decomposition theorem over* $\Gamma(M, E_2)$. *That is to say, we put the* L^2 *norm on* $\Gamma(M, E_i)$ *and complete these spaces. We denote* $\Gamma_2(M, E_i)$ *for these completed hilbert spaces. Let* $\mathbf{H}$=$\{u; u \in \Gamma(M, E_2), \overline{\partial}_1^* u = 0, \overline{\partial}_2 u = 0\}$. *Then, there is a Neumann type operator* N *from* $\Gamma_2(M, E_2)$ *to itself and the harmonic operator* H *from* $\Gamma_2(M, E_2)$ *to* $\mathbf{H}$ *satisfying;*

(1) NH=HN=0,

(2) for $u \in \Gamma_2(M, E_2)$, $u = Hu + \overline{\partial}_2^* \overline{\partial}_2 Nu + \overline{\partial}_1 \overline{\partial}_1^* Nu$.

(3) $\|Nu\| \leq c\|u\|''$, *for* $u \in \Gamma_2(M, E_2)$, *where* c *is a positive constant, independent of* u.

For the norm $\|\ \|''$, see [A1].

3 The new complex inspired by Rumin

Let

$$H_0 = \{v : v \in \Gamma(M, T'), (\overline{\partial}_{T'} v)_{F \otimes ({}^0T'')^*} = 0\}. \tag{10}$$

Instead of (7), we introduce the following differential complex.

$$0 \longrightarrow H_0 \xrightarrow{\overline{\partial}_{T'}} \Gamma(M, E_1) \xrightarrow{\overline{\partial}_1} \Gamma(M, E_2). \tag{11}$$

By the definition of H_0, $\overline{\partial}_{T'} u$ is in $\Gamma(M, {}^0T' \otimes ({}^0T'')^*)$ (because $F \otimes ({}^0T'')^*$) term vanishes). And $\overline{\partial}_{T'}^{(1)} \overline{\partial}_{T'} u = 0$ is obvious. So, (9) makes sense as a differential complex. We have to explain what $H_0$is like. First, we note that H_0 does not come from the space of C^∞ sections of a vector bundle over M. H_0 is a graph of the following first order differential operator. For

$u \cdot \xi$ in $\Gamma(M,F)$, where u is a C^∞ function on M, we set an element ψ_u of $\Gamma(M,{}^0T')$ by:

$$(\overline{\partial}_{T'}(\psi_u + u\cdot\xi))_{F\otimes({}^0T'')^*} = 0, \tag{12}$$

that is to say,

$$[X,\psi_u]_F + (Xu)\xi + u[X,\xi]_F = 0, \quad \text{for } X \in \Gamma(M,{}^0T''). \tag{13}$$

This is possible. Since our CR structure is strongly pseudo convex, ψ_u is uniquely determined. Hence we have a first order differetial operator from $\Gamma(M,F)$ to H_0 defined by; for $u\cdot\xi$ in $\Gamma(M,F)$, $\psi_u + u\cdot\xi$ in H_0, and so we have a differeital complex.

$$\begin{array}{ccccccc} & & \Gamma(M,F) & & & & \\ & & {\scriptstyle\rho}\big\downarrow & & & & \\ 0 & \longrightarrow & H_0 & \xrightarrow{\overline{\partial}_{T'}} & \Gamma(M,E_1) & \xrightarrow{\overline{\partial}_1} & \Gamma(M,E_2). \end{array}$$

We note that ψ_u includes a first derivative of u. We see this more explicitly. By using a moving frame $\{e_1^\lambda, e_2^\lambda\}$ of ${}^0T''|_{U_\lambda}$, satisfying:

$$-\sqrt{-1}[e_i^\lambda, \overline{e}_j^\lambda]_F = \delta_{ij}\xi, \tag{14}$$

where $[e_i^\lambda, \overline{e}_j^\lambda]_F$ means the F part of $[e_i^\lambda, \overline{e}_j^\lambda]$ according to (2.4), we write down ψ_u. Set

$$\psi_u = \psi_1^\lambda \overline{e}_1^\lambda + \psi_2^\lambda \overline{e}_2^\lambda. \tag{15}$$

Then,

$$[e_i^\lambda, \psi_1^\lambda\overline{e}_1^\lambda + \psi_2^\lambda\overline{e}_2^\lambda]_F + (e_i^\lambda u)\xi + u[e_i^\lambda,\xi]_F = 0, \tag{16}$$

that is to say,

$$\sqrt{-1}\psi_i^\lambda\xi + (e_i^\lambda)\xi + u[e_i^\lambda,\xi]_F = 0, \ i = 1,2. \tag{17}$$

So

$$\psi_i^\lambda = \sqrt{-1}e_i^\lambda u + \text{0-th order term of } u. \tag{18}$$

We set a second order linear differential operator D from $\Gamma(M,F)$ to $\Gamma(M,E_1)$, by: for $u\cdot\xi$, where u is a C^∞ function,

$$D(u\cdot\xi)=\overline{\partial}_{T'}(\rho(u\cdot\xi)). \tag{19}$$

We would like to explain about this second order differential operator and Rumin's one. For this, we set a C^∞ vector bundle decomposition over complex valued differential k forms(here we write this space by $\Lambda^k(C\otimes TM)^*$) as follows.

$$\begin{aligned}\Lambda^k(C\otimes TM)^* &= \sum_{r+s=k-1,r,s\geq 0}\theta\wedge\wedge^r({}^0T')^*\wedge\wedge^s({}^0T'')^* \\ &+\sum_{r+s=k,r,s\geq 0}\theta\wedge\wedge^r({}^0T')^*\wedge\wedge^s({}^0T'')^*.\end{aligned} \tag{20}$$

Here θ is defined by:

$$\begin{cases}\theta(\xi)=1,\\ \theta(X)=0,\quad X\in{}^0T',\\ d\theta(\xi,X)=0\quad X\in{}^0T'.\end{cases}$$

(if necessary, we change ξ). By using this decomposition, we introduce; for $i\geq 1$,

$$\begin{aligned}F^{n-2,i}=\{u:u\in\Gamma(M,\quad\theta\quad&\wedge\wedge^{n-2}({}^0T')^*\wedge\wedge^i({}^0T'')^*),\\ &(du)_{\wedge^{n-1}({}^0T')^*\wedge\wedge^{i+1}({}^0T'')^*)}=0\}.\end{aligned} \tag{21}$$

Here $(du)_{\wedge^{n-1}({}^0T')^*\wedge\wedge^{i+1}({}^0T'')^*)}$ means the $\wedge^{n-1}({}^0T')^*\wedge\wedge^{i+1}({}^0T'')^*)$ part of du according to the above decomposition. For $i=0$, we set

$$\Gamma(M,\wedge^{n-1}(T')^*). \tag{22}$$

Then we have a complex version of the Rumin complex

$$\Gamma(M,\wedge^{n-1}(T')^*)\xrightarrow{D}\Gamma(M,F^{n-2,1})\xrightarrow{d''}\Gamma(M,F^{n-2,2}).$$

We note that: if the canonical line bundle is trivial, this is nothing but our new complex.

Our main result is as follows.

Theorem 7 *For this complex, a subelliptic estimate on $\Gamma(M,F^{n-2,1})$ holds (see [A-G-L[2]]).*

And so, this assures that:

Theorem 8 *The following family is parametrized by a finite dimensional analytic space.*

$$\begin{cases} P(\phi) = 0, \\ \overline{\partial}^*_{T'}\phi = 0, \ \textit{for } \phi \in \Gamma(M, E_1). \end{cases}$$

Here $\overline{\partial}^*_{T'}$ means the adjoint operator in the our setting(not the standard one)(see [A-G-L[2]]). And this family is parametrized by a complex analytic space locally at o, complex analytically and versal in the sence of Kuranishi.

We note that the linear term of these non-linear equations is

$$\begin{cases} \overline{\partial}_1\phi = 0, \\ \overline{\partial}^*_{T'}\phi = 0, \ \text{for } \phi \in \Gamma(M, E_1). \end{cases}$$

This is the dual of the Dirac operator in our setting.

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